# Star Formation

# Q & A Session 16.06.2020

The Initial Mass Function

## The Salpeter law

Assume the Salpeter equation describes stars formed in a cluster with masses between $M_l$ and $M_u >> M_l$. Write down and solve the integrals that give

1. the number of stars
2. their total mass
3. the total luminosity assuming $L = L_{\odot}(M/M_{\odot})^{3.5}$

Explain why the number and mass of stars depend mainly on the mass $M_l$ of the smallest stars, while the luminosity depends on $M_u$, the mass of the largest stars.

$$\frac{dn}{dm} = \xi(m) = \xi_0 M^{-2.35} \quad (1)$$

In[289]:=
```
Integrate[M^-2.35, {M, 0, ∞}]
```

... Integrate: Integral of $\frac{1}{M^{2.35}}$ does not converge on {0, ∞}.

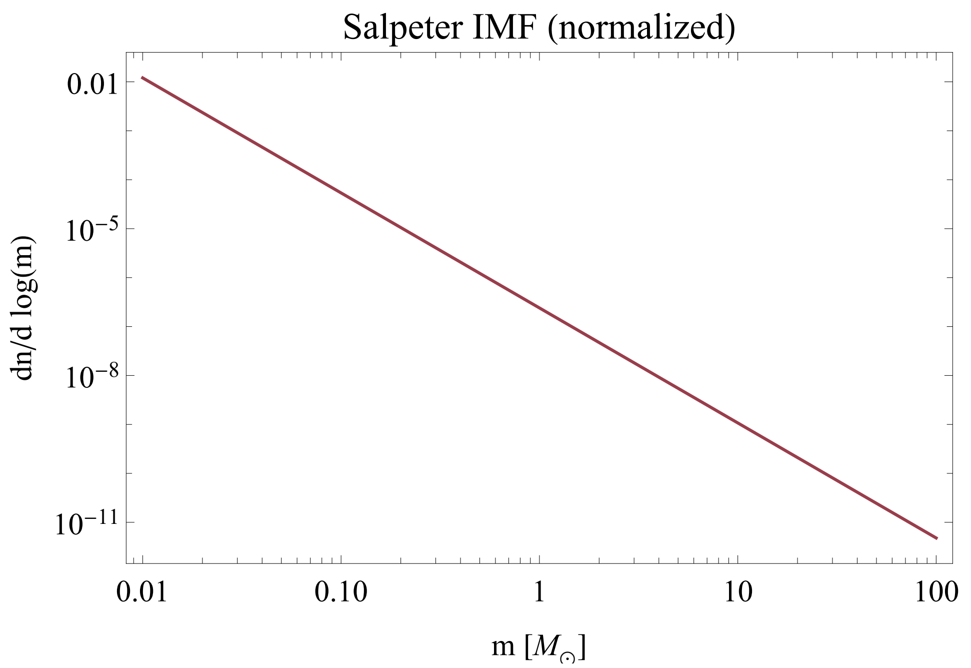
Out[289]= $\int_0^{\infty} \frac{1}{M^{2.35}} dM$

In[273]:=
```
salpeter[m_, ξ0_ : 1] := ξ0 m^-2.35
```

In[278]:=
```
Ainv = NIntegrate[salpeter[m], {m, 10^-5, 1000}]
Plot[salpeter[m, Ainv^-1], {m, 10^-2, 100},
 ScalingFunctions → {"Log", "Log"}, FrameLabel → {"m [M_⊙]", "dn/d log(m)"},
 BaseStyle → {FontSize → 16, FontFamily → "Times New Roman"},
 PlotLabel → "Salpeter IMF (normalized)"]
```

Out[278]= $4.16549 \times 10^{6}$

Out[279]=




1) number of stars

In[257]:=
```
Integrate[ξ0 M^-2.35, {M, M1, Mu}, GenerateConditions → False]
```

Out[257]= $\left(\frac{0.740741}{M_l^{1.35}} - \frac{0.740741}{M_u^{1.35}}\right) \xi_0$

$$\int \xi(M)\, dM = \int_{M_l}^{M_u} \xi_0 M^{-2.35}\, dM = \left(\frac{0.740741}{M_l{}^{1.35}} - \frac{0.740741}{M_u{}^{1.35}}\right) \xi_0 \tag{2}$$

2) mass: We'll need to multiply the integrated function by the mass of the star, to convert our expression for the number of stars into an expression for the sum of their masses.

$$\int \xi(M)\, M dM = \int_{M_l}^{M_u} \xi_0 M M^{-2.35}\, dM = \int_{M_l}^{M_u} \xi_0 M^{-1.35}\, dM = \left(\frac{2.85714}{M_l^{0.35}} - \frac{2.85714}{M_u^{0.35}}\right) \xi_0 \tag{3}$$

In[255]:=
```
Integrate[ξ0 M^-1.35, {M, M1, Mu}, GenerateConditions → False]
```

Out[255]= $\left(\frac{2.85714}{M_l^{0.35}} - \frac{2.85714}{M_u^{0.35}}\right) \xi_0$

The average stellar mass is

$$\frac{\int \xi(M)\, M\, dM}{\int \xi(M)\, dM} = \langle M \rangle \tag{4}$$

In[281]:= $\frac{\text{Integrate}[\xi_0 M^{-1.35}, \{M, 0.1, 100\}, \text{GenerateConditions} \to \text{False}]}{\text{Integrate}[\xi_0 M^{-2.35}, \{M, 0.1, 100\}, \text{GenerateConditions} \to \text{False}]}$

Out[281]= 0.351369

This number depends on the limits of integration

In[291]:= $\frac{\text{Integrate}[\xi_0 M^{-1.35}, \{M, 0.05, 1000\}, \text{GenerateConditions} \to \text{False}]}{\text{Integrate}[\xi_0 M^{-2.35}, \{M, 0.05, 1000\}, \text{GenerateConditions} \to \text{False}]}$

Out[291]= 0.186834

c) luminosity: we use $L \propto L_\odot (M/M_\odot)^{3.5}$, so

$$\int \xi(M)\, L(M)\, dM = \int_{M_l}^{M_u} \xi_0\, L_\odot\, M^{3.5} M^{-2.35}\, dM = \int_{M_l}^{M_u} \xi_0\, L_\odot\, M^{1.15}\, dM = L_\odot \left(-0.465116\, M_l^{2.15} + 0.465116\, M_u^{2.15}\right) \xi_0 \quad (5)$$

In[258]:= Integrate[$\xi_0$ L0 $M^{1.15}$, {M, $M_l$, $M_u$}, GenerateConditions → False]

Out[258]= L0 $\left(-0.465116\, M_l^{2.15} + 0.465116\, M_u^{2.15}\right) \xi_0$

The expressions for the number and mass have negative exponents on the mass limits, so the result will be dominated by the *smaller* number, $M_l$, rather than $M_u$. However, the luminosity expression has positive exponents on the masses, so the larger $M_u$ dominates.

The average luminosity is

$$\frac{\int \xi(M)\, L(M)\, dM}{\int \xi(M)\, dM} = \langle L \rangle \quad (6)$$

In[287]:= $\frac{\text{Integrate}[\xi_0 L_{\text{"}\odot\text{"}} M^{3.5} M^{-2.35}, \{M, 0.1, 100\}, \text{GenerateConditions} \to \text{False}]}{\text{Integrate}[\xi_0 M^{-2.35}, \{M, 0.1, 100\}, \text{GenerateConditions} \to \text{False}]}$

Out[287]= 559.672 $L_\odot$

In[296]:= $\frac{\text{Integrate}[\xi_0 L_{\text{"}\odot\text{"}} M^{3.5} M^{-2.35}, \{M, 0.1, 1000\}, \text{GenerateConditions} \to \text{False}]}{\text{Integrate}[\xi_0 M^{-2.35}, \{M, 0.1, 1000\}, \text{GenerateConditions} \to \text{False}]}$

Out[296]= 79 049.1 $L_\odot$

Taking $M_l = 0.3\, M_\odot$ and $M_u >> 5\, M_\odot$, show that only 2.2% of all stars have $M > 5\, M_\odot$, while these account for 37% of the mass

In[259]:= NumAllStars = Integrate[$\xi_0$ $M^{-2.35}$, {M, 0.3, 100 000}, GenerateConditions → False]

Out[259]= 3.76316 $\xi_0$

In[260]:= NumMLarger5Stars = Integrate[$\xi_0$ $M^{-2.35}$, {M, 5, 100 000}, GenerateConditions → False]

Out[260]= 0.0843444 $\xi_0$

In[261]:= $\frac{\text{NumMLarger5Stars}}{\text{NumAllStars}}$

Out[261]= 0.0224132

In[262]:=
```
MassAllStars = Integrate[ξ0 M^-1.35, {M, 0.3, 100000}, GenerateConditions → False]
```
Out[262]= $4.3037\ \xi_0$

In[263]:=
```
MassMLarger5Stars = Integrate[ξ0 M^-1.35, {M, 5, 100000}, GenerateConditions → False]
```
Out[263]= $1.57584\ \xi_0$

In[264]:=
```
MassMLarger5Stars / MassAllStars
```
Out[264]= 0.366158

Thus the total mass of the high-mass stars is ~0.37 of the total cluster stellar mass.

The Pleiades cluster has $M \sim 800\ M_\odot$ show that it has about 700 stars.

Knowing the total mass we can derive the normalization constant $\xi_0$

In[267]:=
```
AllMassPleiades = Integrate[ξ0 M^-1.35, {M, 0.3, 100000}, GenerateConditions → False]
```
Out[267]= $4.3037\ \xi_0$

In[268]:=
```
Solve[AllMassPleiades == 800, ξ0]
```
Out[268]= $\{\{\xi_0 \to 185.886\}\}$

Inserting in the number of stars integral:

In[269]:=
```
Integrate[ξ0 M^-2.35, {M, 0.3, 100000}, GenerateConditions → False] /. %[[1]]
```
Out[269]= 699.52

Taking $M_u = 10\ M_\odot$ show that the few stars with $M > 5\ M_\odot$ contribute nearly 80% of the light.

In[270]:=
```
totalLight = Integrate[ξ0 L0 M^1.15, {M, 0.3, 10}, GenerateConditions → False]
```
Out[270]= $65.6645\ L0\ \xi_0$

In[271]:=
```
heavyStarLight = Integrate[ξ0 L0 M^1.15, {M, 5, 10}, GenerateConditions → False]
```
Out[271]= $50.8965\ L0\ \xi_0$

In[272]:=
```
heavyStarLight / totalLight
```
Out[272]= 0.7751

so about 78% of all the stellar light comes from massive stars with $M > 5\ M_\odot$

# The Origin of Brown Dwarfs

For the purposes of this problem, we will define a brown dwarf as any object whose mass is below $M_{BD} = 0.075\ M_\odot$, the hydrogen burning limit. We would like to know if these could plausibly be produced via turbulent fragmentation, as appears to be the case for stars.

## a) Brown Dwarf fraction

For a Chabrier (2005) IMF (see Chapter 2, equation 2.3), compute the fraction $f_{BD}$ of the total mass of stars produced that are brown dwarfs.

The Chabrier IMF is

$$\frac{dn}{d\log m} = \xi(m) = \begin{cases} A\exp\left[-\frac{(\log m - \log m_c)^2}{2\sigma^2}\right], & m < 1.0\,M_\odot \\ B\,(m/M_\odot)^{-x}, & m > 1.0\,M_\odot \end{cases} \tag{7}$$

where $m_c = 0.22\,M_\odot$, $\sigma = 0.57$, $x = 1.3$, $A$ is a normalization constant, and the fact that $\xi(m)$ is continuous at $m = 1\,M_\odot$ implies that

$$B = A\exp\left[-\frac{\log(m_c/M_\odot)^2}{2\sigma^2}\right] \tag{8}$$

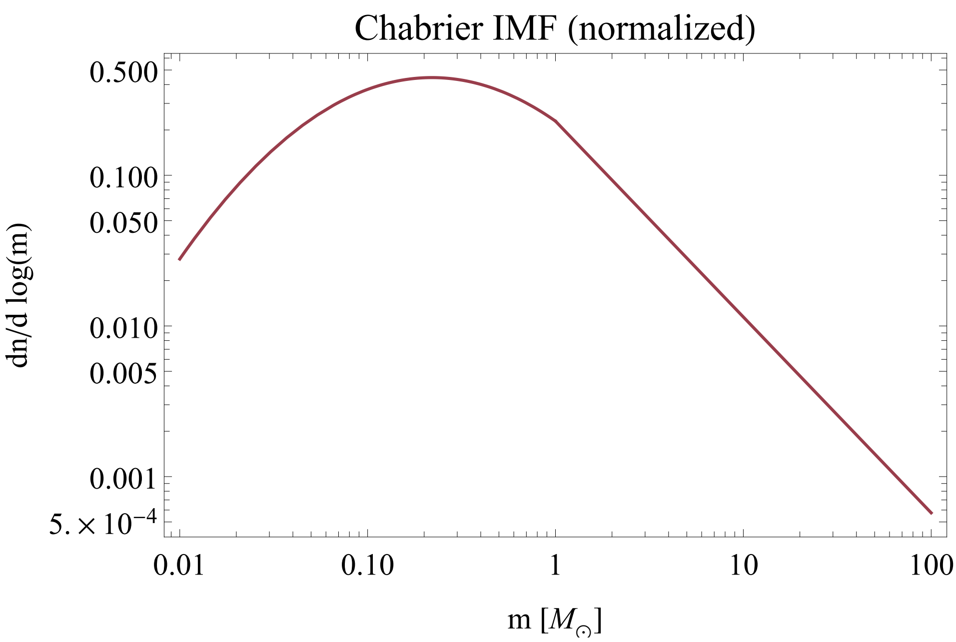
```
In[297]:= chabrier[m_, A_ : 1, B_ : 1] := Piecewise[{{A Exp[-(Log10[m] - Log10[0.22])^2/(2 × 0.57^2)], m ≤ 1.},
    {B Exp[-(Log10[0.22])^2/(2 × 0.57^2)] (m)^-1.3, m > 1.}}]
```

Let's look at the normalization constant first:

```
In[209]:= Ainv = NIntegrate[chabrier[m], {m, 10^-5, 1000}]
Out[209]= 2.24601

In[210]:= NIntegrate[chabrier[m, Ainv^-1], {m, 10^-5, 1000}]
Out[210]= 1.
```

Out[212]=




To compute the fraction of mass in brown dwarfs, $m < m_{BD} = 0.075\,M_\odot$ we simply evaluate the integral of $\xi(m)$ over all masses below $m_{BD}$ and divide by the integral over all masses, i.e.

$$f_{bd} = \frac{\int_{m_{min}}^{m_{BD}} \xi(m)\,dm}{\int_{m_{min}}^{m_{max}} \xi(m)\,dm} \tag{9}$$

Note that we want to integrate with respect to $m$ and not $\log(m)$, because

$$\int \frac{dn}{d\log m}\,dm \propto \int \frac{dn}{dm}\,m\,dm \tag{10}$$

is the mass, which is what we want. The integrals can be evaluated analytically in terms of error functions, but it is more convenient just to evaluate them numerically from this point.

In[213]:=
```
Integrate[chabrier[m, Ainv^-1], m]
```

Out[213]=
$$\int \left( \begin{cases} 0.445235\, e^{-1.53894 \left(0.657577+\frac{\text{Log}[m]}{\text{Log}[10]}\right)^2} & m \le 1. \\ \frac{0.22887}{m^{1.3}} & m > 1. \\ 0 & \text{True} \end{cases} \right) dm$$

Using $m_{min}$ = 0 and $m_{max}$ = 120 $M_\odot$

In[216]:=
```
int1 = Integrate[chabrier[m, Ainv^-1], {m, 0, 0.075}]
```

Out[216]= 0.0125714

In[217]:=
```
int2 = Integrate[chabrier[m, Ainv^-1], {m, 0, 120}]
```

Out[217]= 0.914612

In[218]:=
```
fBD = int1/int2
```

Out[218]= 0.013745

## b) Minimum density

In order to collapse the brown dwarf must exceed the Bonnor-Ebert mass. Consider a molecular cloud of temperature 10 K. Compute the minimum ambient density $n_{min}$ that a region of the cloud must have in order for the thermal pressure to be such that the Bonnor-Ebert mass is less than the brown dwarf mass.

The Bonnor-Ebert mass is

$$M_{BE} = 1.18 \frac{c_s^4}{\sqrt{G^3 P}} = 1.18 \frac{c_s^3}{\sqrt{G^3 \rho}} \tag{11}$$

where we have used $P = \rho c_s^2$. Rearranging for $\rho$, we have

$$\rho = \frac{(1.18\, c_s^3)^2}{G^3 M_{BE}^2} \tag{12}$$

Evaluating this for a gas with $\mu = 3.9 \times 10^{-24}\, g\, \text{cm}^{-3}$, we have $c_s = \sqrt{k_B T/\mu} = \sqrt{1.38 \times 10^{-16} \frac{10}{3.9 \times 10^{-24}}} = 18\,810.8\ \text{cm}\, s^{-1}$, or 0.19 km $s^{-1}$. Inserting 0.075 $M_\odot$ as $M_{BE}$ we find for $\rho$

$$\rho = \frac{(1.18\, c_s^3)^2}{G^3 M_{BE}^2} = \frac{(1.18 \times 18\,810.8^3)^2}{(6.67 \times 10^{-8})^3\, (0.075 \times 2 \times 10^{33})^2} = 9.23948 \times 10^{-18}\ \text{g cm}^{-3} \tag{13}$$

This corresponds to $n_{min} = \rho/\mu = \frac{9.23948 \times 10^{-18}}{3.9 \times 10^{-24}} = 2.3691 \times 10^6\ \text{cm}^{-3}$

In[227]:=
```
nmin = 2.3690969076202386`*^6;
```

## c) The cluster IC348

Assume the cloud has a lognormal density distribution; the mean density is $\bar{n}$ and the Mach number is $\mathcal{M}$. Plot a curve in the $(n, \mathcal{M})$ plane along which the fraction of the mass at densities above $n_{min}$ is equal to $f_{BD}$. Does the gas cloud that formed the cluster IC 348 ($n \approx 5 \times 10^4\ \text{cm}^{-3}$, $\mathcal{M} \approx 7$) fall into the part of the plot where the mass fraction is below or above $f_{BD}$?

First we want to derive an expression for the fraction of the mass above a given density. For a lognormal mass distribution,

$$\frac{dP}{dx} = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\overline{x})^2}{2\sigma_x^2}\right] \tag{14}$$

where $x = \ln(\rho/\overline{\rho})$, we can obtain this by integrating

$$f(>x_0) = \int_{x_0}^{\infty} \frac{dP}{dx} dx = \int_{x_0}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\overline{x})^2}{2\sigma_x^2}\right] \tag{15}$$

In[226]:=
```
Integrate[1/Sqrt[2 π σ^2] Exp[-(x - xmean)^2/(2 σ^2)], {x, x0, ∞}, GenerateConditions → False]
```

Out[226]=
$$\frac{\frac{1}{\sqrt{\frac{1}{\sigma^2}}} - \sigma\,\mathrm{Erf}\left[\frac{x0-xmean}{\sqrt{2}\,\sigma}\right]}{2\sqrt{\sigma^2}}$$

Expressed in terms of the complementary error function this is

$$f(>x_0) = \int_{x_0}^{\infty} \frac{dP}{dx} dx = \frac{1}{2}\,\mathrm{erfc}\left(\frac{x_0-\overline{x}}{\sqrt{2}\,\sigma}\right) \tag{16}$$

For a lognormal turbulent density distribution, we have $\sigma_x \approx \sqrt{\ln(1+\mathcal{M}^2/4})$ and $\overline{x} = \sigma_x^2/2$. The curve we want is the one defined implicitly by the equation $f(>x0) = f_{BD}$ with $x_0 = n_{min}/\overline{n}$. Thus we wish to solve

$$\frac{1}{2}\,\mathrm{erfc}\left(\frac{\ln(n_{min}/\overline{n}) - \ln(1+\mathcal{M}^2/4)/2}{\sqrt{2\ln(1+\mathcal{M}^2/4)}}\right) = f_{bd} \tag{17}$$

For a given $\overline{n}$ we can find the respective $\mathcal{M}$

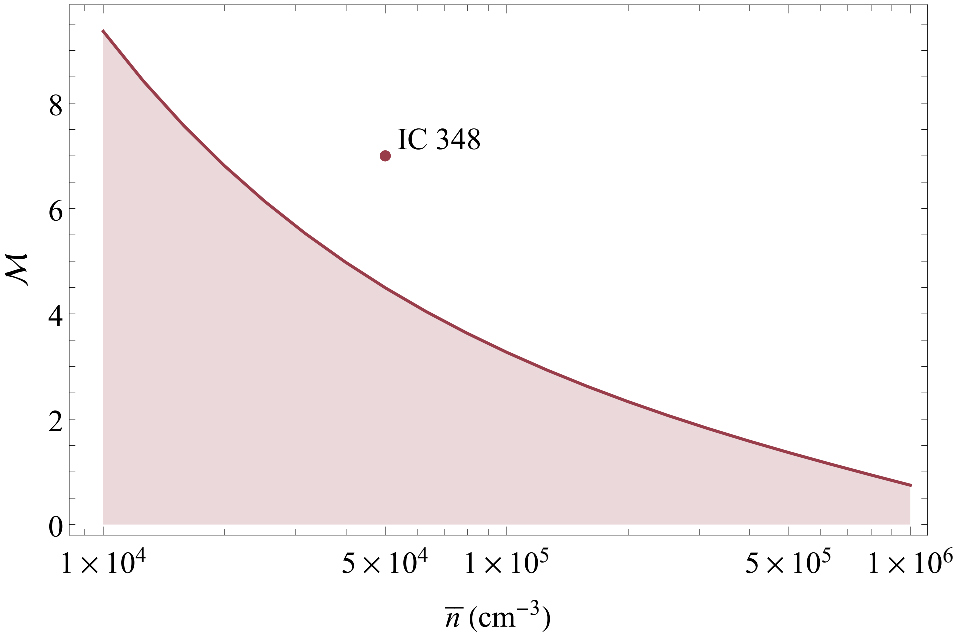
In[242]:=
```
FindRoot[1/2 Erfc[(Log[nmin/10^4] - Log[1 + M^2/4]/2)/Sqrt[2 Log[1 + M^2/4]]] - fBD, {M, 1}]
```

Out[242]= {M → 9.365}

In[243]:=
```
res = Table[{
   10^nmean, First[FindRoot[1/2 Erfc[(Log[nmin/10^nmean] - Log[1 + M^2/4]/2)/Sqrt[2 Log[1 + M^2/4]]] - fBD, {M, 1}]][[
    2]]}, {nmean, 4, 6, .1}]
```

Out[243]=
```
{{10000., 9.365}, {12589.3, 8.41674}, {15848.9, 7.57089},
 {19952.6, 6.81499}, {25118.9, 6.13807}, {31622.8, 5.53052},
 {39810.7, 4.98387}, {50118.7, 4.4907}, {63095.7, 4.04442}, {79432.8, 3.63924},
 {100000., 3.27003}, {125893., 2.93223}, {158489., 2.62178}, {199526., 2.33504},
 {251189., 2.06874}, {316228., 1.81991}, {398107., 1.58582}, {501187., 1.36392},
 {630957., 1.1518}, {794328., 0.947104}, {1. × 10^6, 0.747488}}
```

Out[251]=




The shaded region is the region where $f(>x_0) < f_{\mathrm{BD}}$. Clearly IC 348 (shown as the red dot in the figure) falls into the region where the mass fraction large enough to create brown dwarfs is larger than the brown dwarf mass fraction.